

# Full support of the Kasteleyn operator associated with a bipartite toroidal graph

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**Abstract.** A perfect matching in a bipartite graph embedded on a torus defines a height function on the graph's faces and an associated height change vector in  $\mathbb{Z}^2$ . These matchings are enumerated by a combination of four evaluations of a bivariate Laurent polynomial, called Kasteleyn operator, whose coefficient of bidegree  $(i, j)$  is, up to the sign, the number of perfect matchings with height change  $(i, j)$ . Therefore the Newton polygon of the Kasteleyn operator is the convex hull of the height change vectors. In this article, we prove that any point with integer coordinates in that polygon is realized by a perfect matching.

## 1 Introduction

Consider the set of doubly periodic lozenge tilings of the plane, for a given period  $\mathcal{L} = (\mathbf{u}|\mathbf{v})B\mathbb{Z}^2$ , where  $B$  is an integer  $2 \times 2$  matrix with nonzero determinant and  $\mathbf{u}, \mathbf{v}$  are the (column) vectors depicted in Figure 1a. For instance, the lozenge tiling shown in Figure 1b has the period  $\mathbb{Z}\langle 3\mathbf{u} + 4\mathbf{v}, -5\mathbf{u} + 4\mathbf{v} \rangle$ .

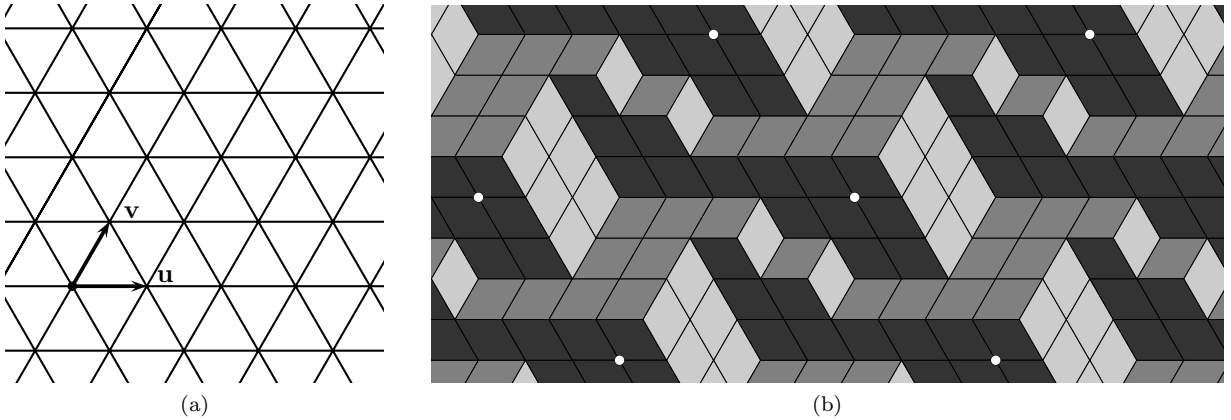


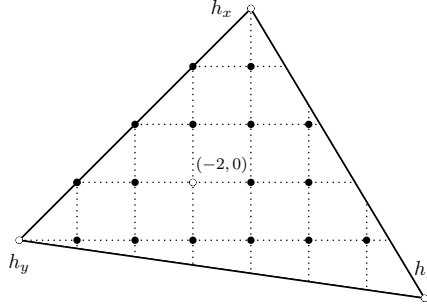
Figure 1

The natural identification of lozenge tilings with stacks of cubes induces a height function that labels the points of  $\mathcal{L}_0 := \mathbb{Z}\langle \mathbf{u}, \mathbf{v} \rangle$ . Being the considered tilings invariant through translations by vectors of the sublattice  $\mathcal{L}$ , such a translation produces a constant increment in the height function. For example, in Figure 1b, the vector  $3\mathbf{u} + 4\mathbf{v}$  reduces the height by 2 and the vector  $-5\mathbf{u} + 4\mathbf{v}$  keeps it unchanged, so we can associate the pair  $(-2, 0)$  to this tiling. Not all the tilings with this period produce the same height change. For instance, the three constant tilings consisting of lozenges of the same type ( $\triangleleft$ ,  $\triangleright$ , and  $\diamond$ , respectively) give the following height changes:  $(1, 9)$ ,  $(-11, -3)$ , and  $(10, -6)$ .

For a general period  $\mathcal{L} \subseteq \mathcal{L}_0$  (and a fixed basis), the possible height change vectors can be determined. Let  $h_x, h_y, h_z$  be the vectors associated to the constant tilings. If, for a  $\mathcal{L}$ -periodic tiling, the amount of lozenges of each type in a fundamental cell of the period is  $(x, y, z)$ , its height change is the following convex combination:  $\frac{\text{vol}(\mathcal{L}_0)}{\text{vol}(\mathcal{L})}(xh_x + yh_y + zh_z) = \frac{1}{|\det(B)|}(xh_x + yh_y + zh_z)$ , which is a point in the triangle determined by  $h_x, h_y$ , and  $h_z$ . Moreover, its difference to any of these vertices lies in  $3\mathbb{Z}^2$ , for the height difference between two points of  $\mathcal{L}_0$  is determined modulo 3, independently of the tiling or the period.

Conversely, for any point  $h$  in the triangle with coordinates in  $h_x + 3\mathbb{Z}^2$ , there exists an  $\mathcal{L}$ -periodic tiling whose height change vector is  $h$ . The goal of this article is to give a proof of this fact in a more general setting,

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which considers perfect matchings in a general bipartite graph embedded on a torus instead of periodic lozenge tilings.

A slight modification of the construction explained above is a particular case of the height function defined by Kenyon et al. [2006] in the general case (see also [Thurston, 1990]). Summing up, a pair of perfect matchings  $(\omega_0, \omega)$  defines a set of circuits (called *transition graph*) on the torus whose homology type  $h_{\omega_0}(\omega) \in \mathbb{Z}^2$  determines the height change vector. Details follow in Subsection 2.1.

For a given bipartite toroidal graph and a fixed base perfect matching  $\omega_0$ , the associated *Newton polygon* is the convex hull of the set of homology types  $h_{\omega_0}(\omega)$  of the transition graphs formed by  $\omega_0$ . It receives this name because it coincides with (a translate of) the Newton polygon of a bivariate Laurent polynomial  $P(w, z)$ , namely, the determinant of a weighted Kasteleyn-Percus matrix (see [Kenyon et al., 2006]). The absolute value of a coefficient of  $P$  is the number of perfect matchings with a corresponding height change. In Section 3 we prove that any point with integer coordinates in the Newton polygon is realized by some perfect matching, so that the support (i.e. the set of occurring monomials) of the Kasteleyn operator is maximal.

## 2 Preparations

In this section we collect some basic facts that will be needed in the proof of our theorem. Firstly, we explain the formalization of the height functions described in the introduction, following [Kenyon, 2009]. Then, we discuss the homology of a set of knots in the torus. Finally, we relate the existence of circuits in circulant digraphs to the visibility of lattice points.

### 2.1 Height functions

In order to fix the class of graphs whose perfect matchings are assigned height functions, let us recall some facts about topological graphs. We refer to [Gross and Tucker, 1987; Mohar, 1988] for a detailed treatment. A graph  $G = (V, E)$  (possibly with loops or multiple edges) is endowed with a natural topology. An *embedding* of a graph  $G$  on a surface  $S$  is a mapping  $i : G \hookrightarrow S$  such that the restriction  $i : G \hookrightarrow i(G)$  is a homeomorphism (with  $i(G)$  endowed with the subspace topology). We usually identify a graph and its image through an embedding, so that  $G \subset S$ . Note that, as we deal with infinite graphs, the usual definition of embedding as an injective and continuous mapping is not equivalent. An embedding is *cellular* if the complement of the graph in the surface is homeomorphic to a disjoint union of open discs.

We use the term *periodic graph* for a cellular embedding of a graph on the plane that is periodic through integer-valued vectors, i.e. the translation by any vector in  $\mathbb{Z}^2$  is a graph automorphism. The existence of a cellular embedding on the plane implies that the graph is locally finite. We will consider bipartite periodic graphs, but will not require them to be connected.

The projection of a periodic graph  $G$  on the torus  $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$  is the embedding of a graph  $\hat{G}$ . This graph is finite, for it is locally finite in a compact surface. Note that the embedding need not be cellular, as is shown in Figure 2.

We use the notation  $\mathcal{M}(G)$  for the set of *perfect matchings* or *1-factors* of a graph  $G$ , i.e.

$$\mathcal{M}(G) := \{\omega \subseteq E \mid \forall v \in V \exists ! e \in \omega : v \in e\}.$$

In the following, we use the terms *matching* and *perfect matching* indistinctly, for we do not consider incomplete matchings. We are interested in those matchings on  $G$  which are compatible with the projection on the torus; or equivalently, matchings on  $\hat{G}$ .

For a bipartite periodic graph, we set a fixed orientation (from the white to the black vertex) at every edge. Given two matchings  $\omega, \omega' \in \mathcal{M}(\hat{G})$ , the *transition graph*  $\omega - \omega'$  is composed by the oriented edges of  $\omega$  and the

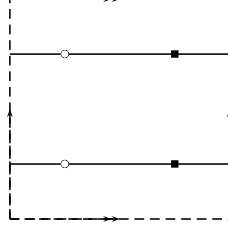


Figure 2

reversed edges of  $\omega'$ . The connected components of a transition graph are directed circuits (*transition cycles*, according to Kasteleyn [1963]) and pairs of vertices bidirectionally linked, which we discard. We assume that the matching set is nonempty and fix a base matching  $\omega_0 \in \mathcal{M}(\hat{G})$ . Let  $\omega \in \mathcal{M}(\hat{G})$  and let us define, by means of the transition graph  $\omega - \omega_0$ , a *height function* on the faces of  $G$ , i.e. the vertices of the geometric dual  $G^*$ . Note that  $G^*$  is connected, although it need not be locally finite (and therefore, it is not necessarily a cellular embedding on the plane). Firstly, we choose a base face and assign the height 0 to it. If there is an edge between two faces which does not occur in the transition graph, both faces get the same height. If the dual edge that goes from a face  $F_1$  to a face  $F_2$  is crossed from left to right by the transition graph, we set  $h(F_2) = h(F_1) + 1$ .

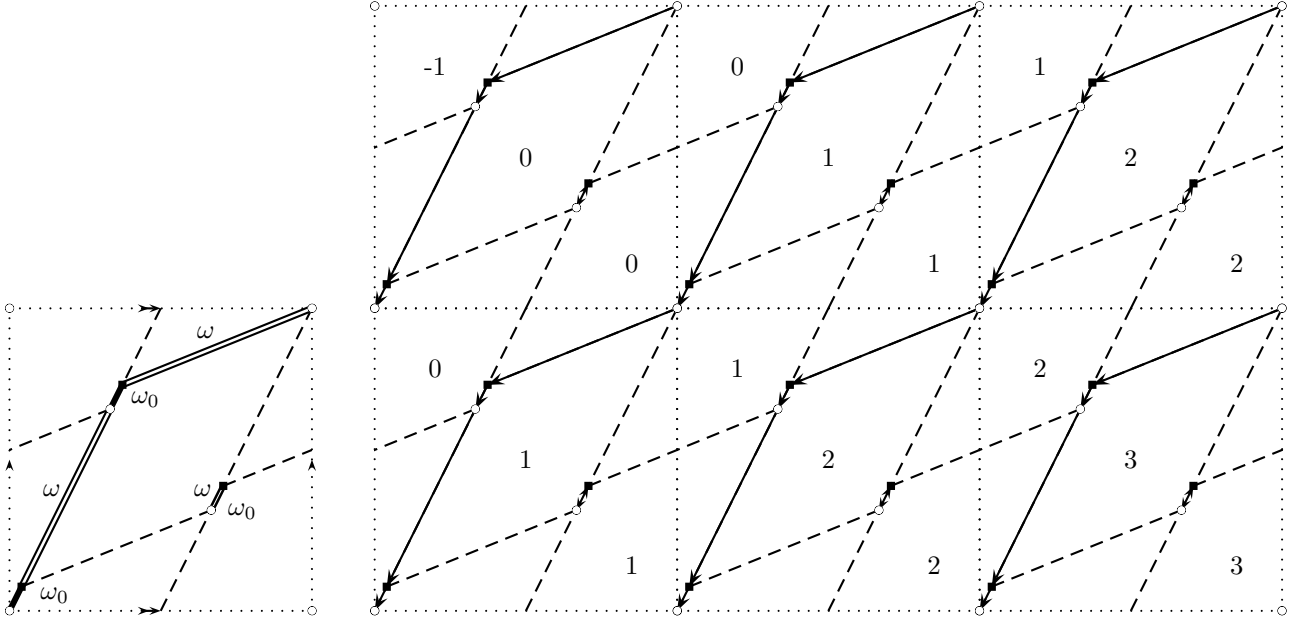


Figure 3

This process consistently defines an integer-valued function on the faces of  $G$  (see an example in Figure 3). To see this, consider the three free Abelian groups generated by the vertex set of  $G$ , the set of edges endowed with the orientation defined above, and the set of (oriented according to the embedding) faces, respectively. The sketch below represents the border homomorphisms  $\partial_i$  and their transposed: the coborder homomorphisms  $\delta_i$ .

$$\begin{array}{ccccc}
 & \delta_0 & & \delta_1 & \\
 \mathbb{Z}\langle V(G) \rangle & \xrightarrow{\quad} & \mathbb{Z}\langle \vec{E}(G) \rangle & \xrightarrow{\quad} & \mathbb{Z}\langle \vec{F}(G) \rangle \\
 & \xleftarrow{\partial_1} & & \xleftarrow{\partial_2} & 
 \end{array}$$

Writing  $\partial^*$  and  $\delta^*$  for the border and coborder homomorphism (respectively) associated with the dual graph,

we have  $\partial_1^* = \delta_1$ ,  $\partial_2^* = \delta_0$ . An element  $\tau^* \in \mathbb{Z}\langle \vec{E}(G^*) \rangle$  defines the homomorphism

$$\begin{aligned} \tau^* = \langle \tau, \cdot \rangle : \mathbb{Z}\langle \vec{E}(G) \rangle &\longrightarrow \mathbb{Z} \\ \omega &\longmapsto \sum_{e \in \vec{E}} \tau(e) \omega(e), \end{aligned}$$

and an analogue construction can be done for elements of  $\mathbb{Z}\langle \vec{F}(G^*) \rangle$ . With this notation, if  $F_1, F_2 \in F(G)$  and  $\tau^*$  is a walk from  $F_1$  to  $F_2$  in the dual graph, we have  $h(F_2) - h(F_1) = \langle \tau, \omega - \omega_0 \rangle$ . Now, if  $\tau_1^*$  and  $\tau_2^*$  are walks in  $G^*$  with the same endpoints, we have to see that  $\langle \tau_1, \omega - \omega_0 \rangle = \langle \tau_2, \omega - \omega_0 \rangle$ . Note that  $\tau_1^* - \tau_2^*$  is a border (there is  $\rho^* \in \mathbb{Z}\langle \vec{F}(G^*) \rangle$  such that  $\partial^*(\rho^*) = \tau_1^* - \tau_2^*$ ) and  $\omega - \omega_0$  is a cycle ( $\partial(\omega - \omega_0) = 0$ ). Therefore  $\langle \tau_1 - \tau_2, \omega - \omega_0 \rangle = \langle \delta(\rho), \omega - \omega_0 \rangle = \langle \partial(\omega - \omega_0), \rho \rangle = 0$ .

On the other hand, if  $F$  and  $F'$  are faces of  $G$  and  $\mathbf{u} \in \mathbb{Z}^2$ , we have  $h(F + \mathbf{u}) - h(F) = h(F' + \mathbf{u}) - h(F')$ . The height function is determined, therefore, by its values on a system of face representatives modulo  $\mathbb{Z}^2$  and the pair of height increments  $\tilde{h}_{\omega_0}(\omega) := (h(F + (1, 0)) - h(F), h(F + (0, 1)) - h(F))$ . As before, it can be shown that the height change vector is determined by the homology type of  $\omega - \omega_0$  in the torus. It is easy to check that, if the homology type of  $\omega - \omega_0$  is  $h_{\omega_0}(\omega) = (a, b) \in \mathbb{Z}^2$ , we have  $\tilde{h}_{\omega_0}(\omega) = (-b, a)$ . For instance, in the example of Figure 3, we have  $\tilde{h}_{\omega_0}(\omega) = (1, -1)$  and  $h_{\omega_0}(\omega) = (-1, -1)$ .

## 2.2 Torus knots

As we have seen, the height change of a matching (with respect to a base matching) can be identified with the homology type of a set of disjoint oriented copies of  $S^1$  on the torus. Let us collect some results on torus knots (whose proofs can be found in [Rolfsen, 1976]) for later use. Note that, unlike usually, we consider directed knots.

**Lemma 1** *Let  $\mathbf{u} = (u_1, u_2) \in \mathbb{Z}^2$ . There exists a torus knot  $c$  with homology type  $\mathbf{u}$  if and only if  $\mathbf{u}$  is a visible lattice point (i.e.  $\mathbf{u} = \mathbf{0}$  or  $\gcd(u_1, u_2) = 1$ ).*

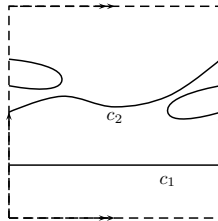
Let  $f \in \text{Aut}(\mathbb{T}^2)$  be a self-homeomorphism on the torus. The functor  $\pi_1$  associates an automorphism of the fundamental group  $\mathbb{Z}^2$  with it:

$$\text{Aut}(\mathbb{T}^2) \xrightarrow{\pi_1} \text{GL}(2, \mathbb{Z}).$$

This mapping is a group epimorphism, in particular, for every group automorphism  $P$  on  $\mathbb{Z}^2$ , there is a self-homeomorphism on the torus whose effect on the homology types of the torus cycles is determined by  $P$ . Moreover:

**Lemma 2** *Let  $c_1, c_2$  be two torus knots with nontrivial homology. Then there is a self-homeomorphism on the torus which maps  $c_1$  into  $c_2$ .*

**Lemma 3** *Let  $c_1, c_2$  be two disjoint torus knots with nontrivial homology types. Then both homology types coincide or are opposite.*



**Proof.** By Lemma 2, we can assume that the homology type of  $c_1$  is  $(1, 0)$ . Therefore  $c_2$  is a knot on the cylinder  $\mathbb{T}^2 \setminus c_1$ , so its homology type must be  $\pm(1, 0)$ . ■

The following result is a direct consequence.

**Lemma 4** *Let  $G$  be a bipartite periodic graph and  $\omega_0, \omega \in \mathcal{M}(\hat{G})$ , such that  $\mathbf{u} = (u_1, u_2) := h_{\omega_0}(\omega) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ . Set  $d := \gcd(u_1, u_2)$ . Then the transition graph consists of circuits with zero homology,  $P$  circuits with homology type  $\frac{1}{d}\mathbf{u}$ , and  $N$  circuits with homology type  $-\frac{1}{d}\mathbf{u}$ , with  $P - N = d$ . In particular, if the transition graph consists of a single circuit,  $\mathbf{u}$  is a visible lattice point.*

Note that if a transition graph  $\omega - \omega_0$  consists of several circuits, the removal of some of them leads to another transition graph  $\omega' - \omega_0$ . Using this remark, we get:

**Lemma 5** Let  $G$  be a bipartite periodic graph and  $\omega_0, \omega \in \mathcal{M}(\hat{G})$ , such that  $\mathbf{u} = (u_1, u_2) := h_{\omega_0}(\omega) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ . Set  $d := \gcd(u_1, u_2)$ . Then there is a matching  $\omega' \in \mathcal{M}(\hat{G})$  such that the transition graph  $\omega' - \omega_0$  consists of one single circuit with homology type  $\frac{1}{d}\mathbf{u}$ .

## 2.3 Circulant digraphs

In the proof of the main result of this article we use a certain class of toroidal graphs whose matchings can be identified with sets of disjoint circuits in directed circulant graphs. Let us recall the definition and some properties of these objects (see the survey by [Bermond et al. \[1995\]](#)).

For integers  $n, j_1, \dots, j_r$  such that  $n \geq 1$ , the *circulant graph*  $C(n; j_1, \dots, j_r)$  is the Cayley graph with vertex group  $\mathbb{Z}/n\mathbb{Z}$  and edges  $\{\{x, x + j_i\} \mid 1 \leq i \leq r\}$ , where the *jumps*  $j_i$  are considered as residue classes modulo  $n$ .

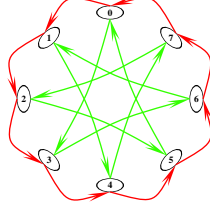


Figure 4:  $\vec{C}(8; 1, 3)$

Note that we are including in the definition circulants with repeated jumps (leading to multiple edges) and zero jumps (leading to loops). We denote by  $\vec{C}(n; j_1, \dots, j_r)$  the corresponding directed graph. For instance, the digraph  $\vec{C}(n; 1, 1)$  consists of a directed cycle of length  $n$  with duplicated arcs.

Let us consider a circulant digraph with two jumps:  $\vec{C}(n; a, b)$ . If a walk consists of  $u$  arcs of the type  $(x, x + a)$  and  $v$  arcs of the other type, we say that its *abelianized* is  $(u, v)$ . If  $x \in \mathbb{Z}/n\mathbb{Z}$  is the initial vertex of such a walk, its terminal vertex is  $x + au + bv$ . We can label, as is done in [\[Wong and Coppersmith, 1974\]](#), a point  $(u, v) \in \mathbb{N}^2$  by  $au + bv$ : the terminal vertex of a walk with abelianized  $(u, v)$  and initial vertex 0.

Note that the set of abelianized closed walks in a connected graph  $\vec{C}(n; a, b)$  is the intersection of a 2-rank integer lattice with volume  $n$  (which we call *circuit lattice*) with  $\mathbb{N}^2$ . For instance, in the case  $\vec{C}(n; 1, b)$ , the circuit lattice is  $\mathbb{Z}\langle(n, 0), (-b, 1)\rangle$ .

We use the term *lattice path* for a finite list of points in  $\mathbb{N}^2$  such that the difference between two consecutive points is  $(1, 0)$  or  $(0, 1)$ . The labelling considered above allows the identification of walks in the digraph with lattice paths. When there are no two distinct points in a lattice path which are congruent modulo the circuit lattice, the associated walk in the digraph is a path (i.e. no vertex is visited twice). When the only pair of congruent points modulo the circuit lattice consists of the endpoints of the lattice path, the associated walk is a circuit.

**Lemma 6** Let  $\Lambda \subseteq \mathbb{Z}^2$  be an 2-rank integer lattice and  $\mathbf{v} \in \Lambda \cap \mathbb{N}^2$ . There is a lattice path from the origin  $\mathbf{0}$  to  $\mathbf{v}$  such that no difference between two distinct path nodes (except  $\mathbf{v} - \mathbf{0}$ ) lies in  $\Lambda$  if and only if  $\mathbf{v}$  is visible in  $\Lambda$  (i.e., the segment joining  $\mathbf{0}$  and  $\mathbf{v}$  does not contain any other point in  $\Lambda$ ) and  $\|\mathbf{v}\|_1 \leq \text{vol}(\Lambda)$ .

**Proof.** The condition  $\|\mathbf{v}\|_1 \leq \text{vol}(\Lambda)$  is necessary: the group  $\mathbb{Z}^2/\Lambda$  has  $\text{vol}(\Lambda)$  elements, a lattice path with length bigger than  $\text{vol}(\Lambda)$  visits at least  $\text{vol}(\Lambda) + 2$  points, and therefore, at least two pairs of them are congruent modulo  $\Lambda$ .

If  $\mathbf{v} = (v_1, v_2)$  is not visible in  $\Lambda$ , the required lattice path cannot exist either. In that case, there is an integer  $d \geq 2$  such that  $\frac{1}{d}\mathbf{v} \in \Lambda$ . Consider a lattice path  $p = (\mathbf{0} = p_0, \dots, p_{\|\mathbf{v}\|_1} = \mathbf{v})$  and its subpaths  $p(i, d)$  with origin at  $p_i$  and length  $\frac{1}{d}\|\mathbf{v}\|_1$ , for  $i = 0, \dots, \frac{d-1}{d}\|\mathbf{v}\|_1$ . Let  $n_i$  be the number of steps of type  $\rightarrow$  that  $p(i, d)$  consists of. If, for some index  $i$ , we have  $n_i = v_1/d$ , the difference of the endpoints of  $p(i, d)$  is  $\frac{1}{d}\mathbf{v} \in \Lambda$ . In other case, since  $n_{i+1} - n_i \in \{1, 0, -1\}$ , we have:  $n_i < v_1/d$ , for every index  $i$ ; or  $n_i > v_1/d$ , for every index  $i$ . Then,

$$v_1 = \sum_{i=0}^{d-1} n(\frac{i}{d}\|\mathbf{v}\|_1) \neq v_1.$$

It can be interesting to compare this implication with the so-called Universal Chord Theorem (see [\[Rolfen, 1976, p. 15\]](#)). For the other, let  $\mathbf{v} \in \Lambda \cap \mathbb{N}^2$  be a point visible in  $\Lambda$  such that  $\|\mathbf{v}\|_1 \leq \text{vol}(\Lambda)$ . Draw the segment  $s$  that joins the origin  $\mathbf{0}$  with  $\mathbf{v}$  and consider the diagonals  $d_i := \{(x, y) \in \mathbb{R}^2 \mid x + y = i\}$ . We define the path

$p = (p_0, p_1, \dots, p_{\|\mathbf{v}\|_1})$  by choosing  $p_i$  as the element of  $d_i$  with integer coordinates that minimizes the distance to  $s \cap d_i$  (see Figure 5a). If the intersection is equally distant from two integer points, we choose the one with a bigger first coordinate. It is easy to see that  $p_{i+1} - p_i \in \{(1, 0), (0, 1)\}$  and  $p$  is a lattice path from  $\mathbf{0}$  to  $\mathbf{v}$ .

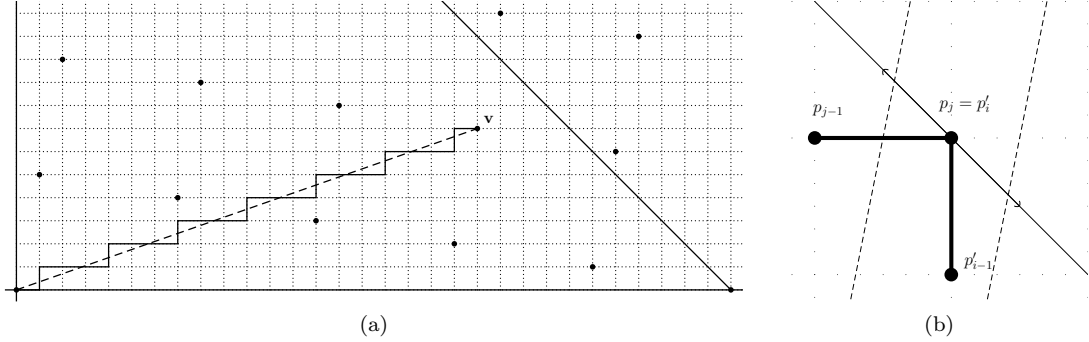


Figure 5

Suppose that  $0 \leq i < j \leq \|\mathbf{v}\|_1$ ,  $j - i < \|\mathbf{v}\|_1$ , and  $p_j - p_i \in \Lambda$ . Note that  $p_j - p_i$  and  $\mathbf{v}$  must be linearly independent, for  $\|p_j - p_i\|_1 = j - i < \|\mathbf{v}\|_1$  and  $\mathbf{v}$  is visible in  $\Lambda$ . Consider the lattice path  $p'$  “parallel” to  $p$  with origin at  $p_j - p_i$ , i.e.  $p'_k := p_k + p_j - p_i$ , for  $k = 0, \dots, \|\mathbf{v}\|_1$ . These paths collide, for  $p'_i = p_j$ . This implies that the image of  $\Lambda$  through the projection onto  $\mathbb{R}\langle(1, -1)\rangle$  parallel to  $s$  has two points with distance smaller than  $\sqrt{2}$  (a pair of points distant exactly  $\sqrt{2}$  is not enough, as we have set up a rule for tie breaking).

As  $\mathbf{v}$  is visible in  $\Lambda$ , there exists  $\mathbf{w} \in \mathbb{Z}^2$  such that  $\Lambda = \mathbb{Z}\langle\mathbf{v}, \mathbf{w}\rangle$ . The projection described above is defined by the equation:

$$\pi(\mathbf{x}) = \mathbf{x} - \frac{x_1 + x_2}{\|\mathbf{v}\|_1} \mathbf{v},$$

so that  $\pi(\Lambda) = \mathbb{Z}\langle\pi(\mathbf{w})\rangle = \mathbb{Z}\langle\frac{\text{vol}(\Lambda)}{\|\mathbf{v}\|_1}(1, -1)\rangle$ , and the distance between two projected lines is at least  $\frac{\text{vol}(\Lambda)}{\|\mathbf{v}\|_1}\sqrt{2}$ . Therefore the paths cannot collide and the proposed lattice path satisfies the requirements. ■

We close this section with a digression, obtaining, as a corollary of the previous result, the well-known characterization of circulant digraphs with two jumps which are Hamiltonian. Curran and Witte [1985] used similar arguments for the study of Hamiltonian paths.

Consider a connected circulant digraph  $\vec{C}(n; a, b)$ , i.e.  $\gcd(n, a, b) = 1$ . A Hamiltonian circuit in this graph corresponds to a lattice path starting at the origin  $\mathbf{0}$ ; ending at a certain  $\mathbf{v}$ ; and such that every element in  $\mathbb{Z}/n\mathbb{Z}$  is the label of exactly one visited point, except 0, which labels both  $\mathbf{0}$  and  $\mathbf{v}$ . In particular,  $\|\mathbf{v}\|_1 = n$ . As the volume of the circuit lattice of  $\vec{C}(n; a, b)$  is  $n$ , Lemma 6 gives the following characterization of Hamiltonian circulant digraphs:

**Corollary 7** *Let  $n$  be a positive integer and  $a, b \in \mathbb{Z}$  such that  $\gcd(a, b, n) = 1$ . Let  $\Lambda := \{(u, v) \in \mathbb{Z}^2 \mid au + bv \in n\mathbb{Z}\}$ . The circulant digraph  $\vec{C}(n; a, b)$  is Hamiltonian if and only if the intersection  $\Lambda \cap \{(x, y) \in \mathbb{N}^2 \mid x + y = n\}$  has a point visible in  $\Lambda$ .*

Note that the intersection of the circuit lattice with the diagonal  $\{x + y = n\}$  and the first quadrant is  $\{(n - i\frac{n}{d}, i\frac{n}{d}) \mid i = 0, \dots, d\}$ , where  $d := \gcd(b - a, n)$ . The point  $(n - i\frac{n}{d}, i\frac{n}{d})$  is visible in  $\Lambda$  if and only if the label of  $(d - i, i)$  has order  $n/d$  in  $\mathbb{Z}/n\mathbb{Z}$ . Therefore Corollary 7 is equivalent to the following characterization (see [Fiol and Yebra, 1988; Locke and Witte, 1999; Yang et al., 1997]), whose proof dates back to [Rankin, 1948].

**Proposition 8** *Let  $n$  be a positive integer and  $a, b \in \mathbb{Z}$  such that  $\gcd(a, b, n) = 1$ . The circulant digraph  $\vec{C}(n; a, b)$  is Hamiltonian if and only if there exist nonnegative integers  $i, j$  such that:*

$$i + j = \gcd(b - a, n) = \gcd(ai + bj, n).$$

### 3 Main result

As we have seen in Section 2.1, given a bipartite periodic graph  $G$  and a matching  $\omega_0$  on its projection  $\hat{G}$ , any matching  $\omega \in \mathcal{M}(\hat{G})$  defines a point  $h_{\omega_0}(\omega) \in \mathbb{Z}^2$ , related to its height function. Consider the associated Newton polygon  $N_{\omega_0}(G)$ : the convex hull of those points. We prove that any point with integer coordinates in this polygon is  $h_{\omega_0}(\omega)$ , for some  $\omega \in \mathcal{M}(\hat{G})$ . This problem can be reduced to visible points in a triangle with one vertex at the origin and the others at visible lattice points (named  $\mathbf{v}$  and  $\mathbf{w}$ ). An example of a graph whose Newton polygon attains this shape is depicted in Figure 6b. It is obtained by projecting the segments  $[0, \mathbf{v}]$  and  $[0, \mathbf{w}]$  onto the torus and substituting an  $\omega_0$ -edge for every crossing.

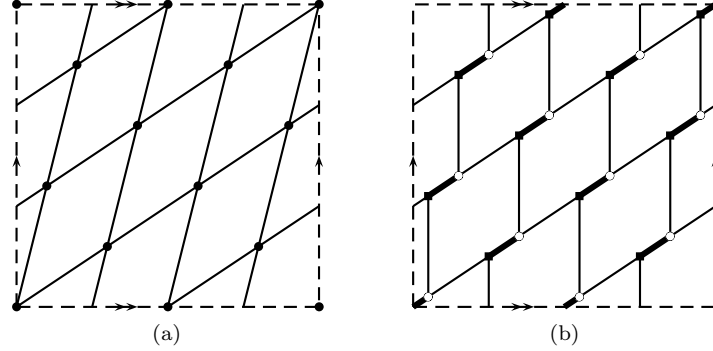


Figure 6:  $\mathbf{v} = (3, 2)$ ,  $\mathbf{w} = (1, 4)$

For integers  $n, r$  such that  $0 \leq r < n$ , we consider the bipartite periodic graph  $B(n, r)$  with black vertices at points  $(i/n, j)$  and white vertices at points  $((2i+1)/2n, j)$ , for integers  $i, j$ . Each white vertex  $((2i+1)/2n, j)$  has the three following neighbours (see Figure 7):  $(\frac{i}{n}, j)$ ,  $(\frac{i+1}{n}, j)$ ,  $(\frac{i+r}{n}, j+1)$ . The edges of the first type constitute a perfect matching  $\omega_0$ , which we fix as base matching.

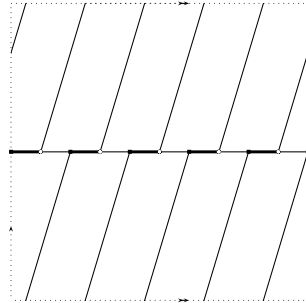


Figure 7: A projection of  $B(5, 2)$

**Proposition 9** *Let  $n, r \in \mathbb{Z}$  such that  $0 \leq r < n$  and  $\gcd(r, n) = 1$ , and consider the (closed) triangle  $T$  with vertices  $(0, 0)$ ,  $(1, 0)$ , and  $(r, n)$ . If  $\mathbf{u}$  is a visible lattice point in  $T$ , there is a matching  $\omega \in \mathcal{M}(\hat{B}(n, r))$  such that  $h_{\omega_0}(\omega) = \mathbf{u}$ .*

**Proof.** Consider the projected graph  $\hat{B}(n, r)$  and identify vertex  $(i/n, j)$  with  $((2i+1)/2n, j)$ , obtaining the circulant graph  $C(n; 1, r)$ . Matchings in  $\hat{B}(n, r)$  correspond with sets of disjoint circuits in  $\vec{C}(n; 1, r)$ , whose circuit lattice is  $\Lambda := \mathbb{Z}\langle(n, 0), (-r, 1)\rangle$ , and the following bijection relates the homology type of a matching to the abelianized of the corresponding set of circuits.

$$\begin{aligned} t: T \cap \mathbb{Z}^2 &\longrightarrow \Lambda \cap \{(x, y) \in \mathbb{N}^2 \mid x + y \leq n\} \\ \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto \begin{pmatrix} n & -r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

Let  $\mathbf{u} = (u_1, u_2) \in \mathbb{Z}^2$  be a visible point in  $T \cap \mathbb{Z}^2$ . By Lemma 6, there is a circuit in  $\vec{C}(n; 1, r)$  whose abelianized is  $t(\mathbf{u})$ . If  $\omega$  is the associated matching, the homology type of  $\omega - \omega_0$  is  $\mathbf{u}$ .  $\blacksquare$



**Theorem 10** *Let  $G$  be a bipartite periodic graph and  $\omega_0 \in \mathcal{M}(\hat{G})$ . We have:*

$$N_{\omega_0}(G) \cap \mathbb{Z}^2 = \{h_{\omega_0}(\omega) \mid \omega \in \mathcal{M}(\hat{G})\}.$$

**Proof.** We have to prove that the set on the left-hand side is contained in the other, which we denote by  $h(G)$ . Let  $\mathbf{u} \in N_{\omega_0}(G) \cap \mathbb{Z}^2$ . It is easy to see that this element is in  $h(G)$  if it is a vertex of  $N_{\omega_0}(G)$  or  $\mathbf{u} = \mathbf{0} = h_{\omega_0}(\omega_0)$ .

Note that, for any three matchings  $\omega_1, \omega_2, \omega \in \mathcal{M}(\hat{G})$ ,  $h_{\omega_1}(\omega) = h_{\omega_1}(\omega_2) + h_{\omega_2}(\omega)$ . Therefore a change in the base matching is equivalent to a translation in the corresponding set  $h(G)$ . Suppose that  $\mathbf{v}, \mathbf{w} \in h(G)$ . Using Lemma 4, any other point with integer coordinates in the segment they define lies also in  $h(G)$ .

By Caratheodory's theorem,  $\mathbf{u}$  lies in the triangle defined by  $\mathbf{0}$  and two vertices of  $N_{\omega_0}(G)$ . Using the previous argument, we can assume w.l.o.g. that  $\mathbf{u}$  is a visible point in the triangle defined by  $\mathbf{0}$  and two visible and linearly independent points  $\mathbf{v}, \mathbf{w} \in h(G)$ . By Lemma 5, there exist  $\omega_1, \omega_2 \in \mathcal{M}(\hat{G})$  with respective homology types  $\mathbf{v}$  and  $\mathbf{w}$ , and such that each of the transition graphs  $\omega_1 - \omega_0$  and  $\omega_2 - \omega_0$  consists of a single circuit.

We know (see the discussion in Subsection 2.2) that there is a self-homeomorphism  $f$  on the torus whose associated automorphism on the first homology group satisfies:

$$\pi_1(f)(\mathbf{v}|\mathbf{w}) = \begin{pmatrix} 1 & r \\ 0 & n \end{pmatrix},$$

where  $0 \leq r < n := |\det(\mathbf{v}|\mathbf{w})|$  and  $\gcd(r, n) = 1$ . Using Lemma 2, another torus self-homeomorphism transforms the transition cycle  $\omega_1 - \omega_0$  into the meridian  $[0, 1] \times \{0\}$ . The set of height change vectors of the transformed graph  $\hat{G}_1$  is  $\pi_1(f)h(G)$ , so we need just prove that any visible point in the triangle defined by  $(0, 0)$ ,  $(1, 0)$ , and  $(r, n)$  is in  $h(G_1)$ .

Now, we transform  $\hat{G}_1$  into another graph  $\hat{H}$  such that  $h(H) \subseteq h(G_1)$ . Consider the intersections of the meridian (image of  $\omega_1 - \omega_0$ ) and the image of the circuit  $\omega_2 - \omega_0$ . Such an intersection cannot be limited to a single point. If this was the case, that point should be a vertex incident to two different edges of  $\omega_0$ , which would

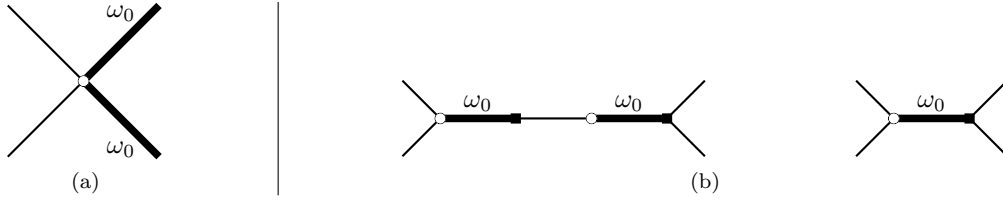


Figure 8

not be a perfect matching (see Figure 8a). Therefore each connected component of the intersection consists of an odd number of edges, which can be reduced to one, as depicted in Figure 8b.

Suppose that in  $G_1$  there are two consecutive (as walking through the meridian) crossings between both considered circuits with the same  $Y$ -coordinate. We can remove (see Figure 9) all those crossings (which happen in

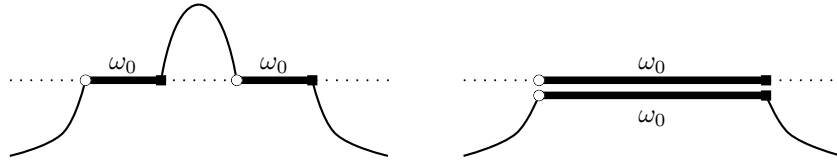


Figure 9

finite number, for the projected graph is finite), obtaining a graph  $H$  homeomorphic (as a planar embedding) to  $B(n, r)$ . Indeed, after the removal process, there are exactly  $n$  crossings in  $\hat{H}$ , and the edges of the second circuit define a permutation of them. As there is no self-intersection, this permutation must be of the type  $x \mapsto x + s$ . We have  $\gcd(s, n) = 1$  ( $s = r$ , indeed), for  $\omega_2 - \omega_0$  consists of a single circuit. Therefore Proposition 9 applies. ■

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